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Smooth approximation of finitely many relativistic point interactions

Walter B Huddell III[1] **and Rhonda J Hughes**[2]

[1] Department of Mathematics, Eastern University, St. Davids, PA 19087, USA
[2] Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010, USA

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Abstract
The purpose of this paper is to show that finitely many relativistic point interactions may be approximated in the strong resolvent sense by sequences of operators with smooth potentials. We consider the entire family of relativistic point interactions, and determine those for which renormalization of the coupling constant occurs when the corresponding potentials are approximated by finitely many local, short-range perturbations of the free Dirac operator.

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1. Introduction

The study of point interactions has a venerable history, beginning with the work of Kronig and Penney [12], and supported by a vast literature that illustrates the wide range of applications and fundamental importance of this concept for physics and mathematics (cf [1, 5–7, 13, 16] for a variety of results in the nonrelativistic case).

Relativistic point interactions (cf [4]) and, more generally, finite-rank singular perturbations of nonsemibounded self-adjoint operators, have been the focus of considerable recent attention. Results on approximation by sequences of operators with regular perturbations have been obtained by Albeverio and Kurasov [2], Albeverio, Koshmanenko, Kurasov, and Nizhnik [3], Koshmanenko [11], and Hughes [8, 9], where renormalization questions are also addressed. In [8], we determined those relativistic point interactions for which renormalization of the coupling constant occurs when the corresponding potentials are approximated by local, short-range perturbations of the free Dirac operator in one dimension. In addition, we found a formula for the renormalization constant for the entire four-parameter family of relativistic point interactions. The purpose of this paper is to extend those results to the case of finitely many relativistic point interactions, including combinations different point interactions at distinct centres.

As in [4], our development begins with consideration of the self-adjoint Dirac operator

$$D_0 = -\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3$$

with domain

$$\mathrm{Dom}(D_0) = H^1(\mathbf{R}) \otimes \mathbf{C}^2,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. The relevant operator in terms of point interactions is obtained by defining the symmetric operator $\hat{H} = D_0$ restricted to

$$\mathrm{Dom}(\hat{H}) = \{\psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2 | \psi(0) = 0\}.$$

$\hat{H}$ has deficiency indices (2, 2), and yields a four-parameter family of self-adjoint extensions. The self-adjoint extensions of interest are those that satisfy the boundary conditions $\psi(0^+) = \Lambda\psi(0^-)$, where

$$\Lambda = \omega \begin{pmatrix} \alpha & \mathrm{i}\beta \\ -\mathrm{i}\gamma & \delta \end{pmatrix},$$

$\omega \in \mathbf{C}$, $|\omega| = 1$ and $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ satisfy $\alpha\delta - \beta\gamma = 1$, (cf [4]). These extensions correspond to point-interactions concentrated at the point $x = 0$ that link the intervals $(-\infty, 0)$ and $(0, \infty)$.

Examples. There are several one-parameter sub-classes of these operators that have received a great deal of attention in the literature:

$$\Lambda_1 = \begin{pmatrix} \cos\theta & -\mathrm{i}\sin\theta \\ -\mathrm{i}\sin\theta & \cos\theta \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} \cosh\theta & -\mathrm{i}\sinh\theta \\ \mathrm{i}\sinh\theta & \cosh\theta \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} 1 & 0 \\ \frac{-\mathrm{i}\alpha}{c} & 1 \end{pmatrix}, \qquad \Lambda_4 = \begin{pmatrix} 1 & \mathrm{i}\beta c \\ 0 & 1 \end{pmatrix}.$$

These correspond to the electrostatic and Lorentz scalar point interactions and the relativistic δ- and δ'-potentials (cf [9, p 429]), respectively.

In [9, theorem 1], it is shown that the class of extensions of $\hat{H}$, which are denoted H^Λ, has the following closed-form expression:

$$H^\Lambda = \mathrm{e}^{-H(x)\otimes A^*} \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\right) \mathrm{e}^{-H(x)\otimes A} + \frac{c^2}{2} \otimes \sigma_3,$$

on

$$D(H^\Lambda) = \{\psi \in L^2(\mathbf{R}) \otimes \mathbf{C}^2 \mid \mathrm{e}^{-H(x)\otimes A}\psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2\},$$

where A is a 2×2 complex matrix for which $\mathrm{e}^A = \Lambda$, and $H(x)$ is the Heaviside function. We will now consider an example associated with Λ_3, to demonstrate that H^Λ is a formal perturbation of D_0.

It is well known in the literature (cf [4, p 162], [8, example 3]) that

$$\Lambda_3 = \begin{pmatrix} 1 & 0 \\ \frac{-\mathrm{i}\alpha}{c} & 1 \end{pmatrix}$$

corresponds to the relativistic δ-potential interaction. Power series methods yield

$$\ln(\Lambda_3) = \Lambda_3 - I = \begin{pmatrix} 0 & 0 \\ \frac{-\mathrm{i}\alpha}{c} & 0 \end{pmatrix} = A_3,$$

which we note is nilpotent. Therefore, by the functional calculus, $\Lambda_3 = \mathrm{e}^{A_3}$. In addition,

$$A_3^* = \begin{pmatrix} 0 & \frac{\mathrm{i}\alpha}{c} \\ 0 & 0 \end{pmatrix}.$$

Therefore, in the case of H^{Λ_3}, we see that

$$\mathrm{e}^{-H(x)\otimes A_3} = \exp\left[\frac{\mathrm{i}\alpha}{c} H(x) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right],$$

and

$$\mathrm{e}^{-H(x)\otimes A_3^*} = \exp\left[\frac{-\mathrm{i}\alpha}{c} H(x) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right].$$

Let $\psi = (\psi_1, \psi_2) \in D(H^{\Lambda_3})$. It is shown in [8, lemma 1] (also, cf [2]), that in the sense of distributions,

$$H^{\Lambda_3}\psi = \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3\right)\psi + \mathrm{i}c \begin{pmatrix} \psi_2(0^+) - \psi_2(0^-) & 0 \\ 0 & \psi_1(0^+) - \psi_1(0^-) \end{pmatrix} \vec{\delta},$$

where $\vec{\delta}$ is a linear map from $H^1(\mathbf{R}) \otimes \mathbf{C}^2$ into $\mathbf{C}^2$ defined by

$$(\vec{\delta}, f) = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix},$$

for $f = (f_1, f_2) \in H^1(\mathbf{R}) \otimes \mathbf{C}^2$. Making use of the boundary conditions, $\psi(0^+) = \Lambda_3\psi(0^-)$, we obtain

$$\begin{aligned} H^{\Lambda_3}\psi &= \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3\right)\psi + \alpha \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{\delta}\psi \\ &= D_0\psi + \alpha \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{\delta}\psi. \end{aligned}$$

2. The main results

We will now extend the results in [8] to the case of finitely many centres. In this case $\hat{H} = D_0$, restricted to

$$\mathrm{Dom}(\hat{H}) = \{\psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2 \mid \psi(x_i) = 0, \text{for } i = 1, \ldots, k\},$$

where $\{x_i\}_{i=1}^k \subset \mathbf{R}$. The deficiency indices of $\hat{H}$ are $(2k, 2k)$, and we consider self-adjoint extensions H^Λ, where $\Lambda = \{\Lambda_i\}_{i=1}^k$ and

$$\Lambda_i = \omega_i \begin{pmatrix} \alpha_i & \mathrm{i}\beta_i \\ -\mathrm{i}\gamma_i & \delta_i \end{pmatrix}, \tag{1}$$

$\omega_i \in \mathbf{C}$, $|\omega_i| = 1$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbf{R}$ satisfy $\alpha_i\delta_i - \beta_i\gamma_i = 1$, for all $i = 1, \ldots, k$. Let

$$\mathrm{Dom}(H^\Lambda) = \left\{\psi \in H^1\left(\mathbf{R} - \{x_i\}_{i=1}^k\right) \otimes \mathbf{C}^2 | \psi\left(x_i^+\right) = \Lambda_i\psi(x_i^-)\right\},$$

and define $H^\Lambda\psi = D_0\psi$ for $\psi \in \mathrm{Dom}(H^\Lambda)$. We will write Λ for $\{\Lambda_i\}_{i=1}^k$.

We now develop a closed-form expression for H^Λ, as was done in the one-centre case. Let $f(z) = \ln(z)$ be a branch of the logarithm on a domain $S \subset \mathbf{C}$, so S does not contain the origin. We note that for all $i = 1, \ldots, k$, Λ_i, considered as an operator on $\mathbf{C}^2$, has finite spectrum $\sigma(\Lambda_i)$ that does not contain 0. Then $f(\Lambda_i) = \ln(\Lambda_i)$ is defined for each i by the Riesz functional calculus [10, 14]. Let $A_i = \ln(\Lambda_i)$. Since $g(z) = \mathrm{e}^z$ is entire, the functional calculus also yields $\mathrm{e}^{A_i} = \Lambda_i$.

Define $H_i(x) = H(x_i - x)$, the translated Heaviside, rotated about the line $x = x_i$, and consider the operator $\sum_{i=1}^k H_i(x) \otimes A_i \in B(L^2(\mathbf{R}) \otimes \mathbf{C}^2)$. Again, by the Riesz functional calculus, $\mathrm{e}^{\sum_{i=1}^k H_i(x)\otimes A_i} \in B(L^2(\mathbf{R}) \otimes \mathbf{C}^2)$. We now give the closed-form expression of H^Λ.

Theorem 1. *Suppose that the matrices* $\{A_i\}_{i=1}^k$ *commute pairwise. Let*

$$\mathrm{Dom}(T) = \left\{\psi \in L^2(\mathbf{R}) \otimes \mathbf{C}^2 \,\middle|\, \mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)}\psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2\right\},$$

and define

$$T\psi = \mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i^*)}\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)}\psi + \frac{c^2}{2}\otimes\sigma_3\psi$$

for $\psi \in \mathrm{Dom}(T)$. *Then* $T = H^\Lambda$.

Proof. We note that T is self-adjoint, so it suffices to prove that $H^\Lambda \subset T$. Let $\psi \in D(H^\Lambda)$. Then $\psi \in H^1\left(\mathbf{R} - \{x_i\}_{i=1}^k\right) \otimes \mathbf{C}^2 \subset L^2(\mathbf{R}) \otimes \mathbf{C}^2$. Now, for each $j \leqslant k$, since the A_j commute,

$$\begin{aligned}
\mathrm{e}^{(-\sum_{i=1}^k H_i(x_j^+)\otimes A_i)}\psi\left(x_j^+\right) &= \mathrm{e}^{(-\sum_{i=1}^j 1\otimes A_i)}\Lambda_j\psi(x_j^-)\\
&= \mathrm{e}^{(-\sum_{i=1}^j 1\otimes A_i)}\,\mathrm{e}^{A_j}\psi(x_j^-)\\
&= \mathrm{e}^{(-\sum_{i=1}^j 1\otimes A_i)+(1\otimes A_j)}\psi(x_j^-)\\
&= \mathrm{e}^{(-\sum_{i=1}^{j-1} 1\otimes A_i)}\psi(x_j^-),
\end{aligned}$$

where one is the function $f(x) \equiv 1$. Also

$$\mathrm{e}^{(-\sum_{i=1}^k H_i(x_j^-)\otimes A_i)}\psi(x_j^-) = \mathrm{e}^{(-\sum_{i=1}^{j-1} 1\otimes A_i)}\psi(x_j^-).$$

Therefore $\mathrm{e}^{(-\sum_{i=1}^k H_i(x_j^+)\otimes A_i)}\psi(x_j^+) = \mathrm{e}^{(-\sum_{i=1}^k H_i(x_j^-)\otimes A_i)}\psi(x_j^-)$, so that $\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)}\psi(x)$ is continuous at x_j. Since $\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)}\psi(x)$ is continuous at x_j, and $\mathrm{e}^{(-\sum_{i=1}^k H_i\otimes A_i)}\psi \in H^1\left(\mathbf{R} - \{x_i\}_{i=1}^k\right) \otimes \mathbf{C}^2$, $\mathrm{e}^{(-\sum_{i=1}^k H_i\otimes A_i)}\psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2$. Hence $\psi \in \mathrm{Dom}(T)$.

To show that $H^\Lambda\psi = T\psi$ for $\psi \in \mathrm{Dom}(H^\Lambda)$, let $\psi \in \mathrm{Dom}(H^\Lambda)$. Then $\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)}\psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2$. In [8, theorem 1], we proved that

$$\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i^*)}\sigma_1\,\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)} = \sigma_1,$$

from which it follows that for almost every x:

$$\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i^*)}\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\mathrm{e}^{(-\sum_{i=1}^k H_i(x)\otimes A_i)}\psi(x) = \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\psi(x).$$

So $H^\Lambda = T$. □

We now consider the problem of smooth approximation in the case of finitely many centres. Let $H_{i,n}(x) = \int_{-\infty}^x h_{i,n}(y)\,\mathrm{d}y$ with $h_{i,n}$ absolutely continuous, non-negative functions with support in $\left(x_i, x_i + \frac{1}{n}\right)$, such that $\int_{-\infty}^{\infty} h_{i,n}(x)\,\mathrm{d}x = 1$ for each i, n. Let $U = \mathrm{e}^{-\sum_{i=1}^k H_i\otimes A_i}$ and define U_n analogously. Then

$$\begin{aligned}
H^{\Lambda_n} &= U_n^*\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)U_n + \frac{c^2}{2}\otimes\sigma_3\\
&= -\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1 + \frac{c^2}{2}\otimes\sigma_3 + \mathrm{i}c\sum_{i=1}^k H'_{i,n}(x)\otimes\sigma_1 A_i
\end{aligned}$$

(cf [8, theorem 1]). We start with a theorem from which the approximation result will quickly follow.

Theorem 2. $U_n \to U$ *and* $U_n^{-1} \to U^{-1}$, *in the strong operator topology.*

Proof. Let $\psi \in L^2(\mathbf{R}) \otimes \mathbf{C}^2$. We proceed with a dominated convergence argument. We have that

$$\left\| \mathrm{e}^{-\sum_{i=1}^{k} H_{i,n} \otimes A_i} \psi - \mathrm{e}^{-\sum_{i=1}^{k} H_i \otimes A_i} \psi \right\|^2 = \int_{-\infty}^{\infty} (f_n(x), f_n(x))_{\mathbf{C}^2} \, \mathrm{d}x$$

where

$$f_n(x) = \mathrm{e}^{-\sum_{i=1}^{k} H_{i,n}(x) \otimes A_i} \psi(x) - \mathrm{e}^{-\sum_{i=1}^{k} H_i(x) \otimes A_i} \psi(x).$$

Now, given $x \in \mathbf{R} - \{x_1, \ldots, x_k\}$, there exists an N such that, for all i, $H_{i,n}(x) = H_i(x)$ for all $n \geqslant N$. Thus $(f_n(x), f_n(x))_{\mathbf{C}^2} = 0$ for $n \geqslant N$ so $(f_n(x), f_n(x))_{\mathbf{C}^2} \to 0$ pointwise.

Also,

$$\begin{aligned} (f_n(x), f_n(x))_{\mathbf{C}^2} &= \left\| \mathrm{e}^{-\sum_{i=1}^{k} H_{i,n}(x) \otimes A_i} \psi(x) - \mathrm{e}^{-\sum_{i=1}^{k} H_i(x) \otimes A_i} \psi(x) \right\|_{\mathbf{C}^2}^2 \\ &\leqslant \left\| \mathrm{e}^{-\sum_{i=1}^{k} H_{i,n}(x) \otimes A_i} - \mathrm{e}^{-\sum_{i=1}^{k} H_i(x) \otimes A_i} \right\|^2 \left\| \psi(x) \right\|_{\mathbf{C}^2}^2 \\ &\leqslant \left(\left\| \mathrm{e}^{-\sum_{i=1}^{k} H_{i,n}(x) \otimes A_i} \right\| + \left\| \mathrm{e}^{-\sum_{i=1}^{k} H_i(x) \otimes A_i} \right\| \right)^2 \| \psi(x) \|_{\mathbf{C}^2}^2. \end{aligned}$$

Both $\left\| \mathrm{e}^{-\sum_{i=1}^{k} H_{i,n}(x) \otimes A_i} \right\|$ and $\left\| \mathrm{e}^{-\sum_{i=1}^{k} H_i(x) \otimes A_i} \right\|$ are bounded by a suitable constant K that is independent of n. Thus we have $(2K)^2 \|\psi(x)\|_{\mathbf{C}^2}^2$ as an $L^1(\mathbf{R})$ bound, and the result follows from the dominated convergence theorem. The proof of the convergence of U_n^{-1} to U^{-1} is identical. $\square$

This leads us to

Theorem 3. *Let*

$$H^{\Lambda_n} = U_n^* \left(-\mathrm{i}c \frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 \right) U_n + \frac{c^2}{2} \otimes \sigma_3,$$

with

$$D(H^{\Lambda_n}) = \{\psi \in L^2(\mathbf{R}) \otimes \mathbf{C}^2 \mid U_n \psi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2\} = H^1(\mathbf{R}) \otimes \mathbf{C}^2.$$

Then H^{Λ_n} converges to H^{Λ} in the strong resolvent sense.

Proof. We will prove strong graph convergence, which is equivalent to strong resolvent convergence. Let $\psi \in D(H^\Lambda)$. Then $U\psi = \phi \in H^1(\mathbf{R}) \otimes \mathbf{C}^2$. So $\psi = U^{-1}\phi$. Let $\psi_n = U_n^{-1}\phi$. Then $U_n \psi_n = U_n U_n^{-1} \phi = \phi$. Thus $\psi_n \in D(H^{\Lambda_n})$, and by theorem 2, $\|\psi_n - \psi\| = \left\| U_n^{-1}\phi - U^{-1}\phi \right\| \to 0$. Moreover,

$$\begin{aligned} \|H^{\Lambda_n}\psi_n - H^\Lambda \psi\| &= \left\| U_n^* \left(-\mathrm{i}c \frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 \right) U_n \psi_n - U^* \left(-\mathrm{i}c \frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 \right) U \psi \right\| \\ &= \left\| U_n^* \left(-\mathrm{i}c \frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 \right) \phi - U^* \left(-\mathrm{i}c \frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 \right) \phi \right\| \to 0 \end{aligned}$$

by theorem 2, since U_n and U are unitary. So $H^{\Lambda_n} \to H^\Lambda$ in the sense of strong graph convergence, and consequently in the strong resolvent sense. $\square$

Next, we consider the question of renormalization of the coupling constants when we pass to the limit as $n \to \infty$. In order to compare this to the result in [2], we first prove the following; here $\vec{\delta}_i$, $i = 1, \ldots, k$ is a linear map from $H^1(\mathbf{R}) \otimes \mathbf{C}^2$ into $\mathbf{C}^2$ defined by

$$(\vec{\delta}_i, f) = \begin{pmatrix} f_1(x_i) \\ f_2(x_i) \end{pmatrix},$$

for $f = (f_1, f_2) \in H^1(\mathbf{R}) \otimes \mathbf{C}^2$.

Lemma 1. *For* $\psi \in \mathrm{Dom}(H^{\Lambda})$,

$$H^{\Lambda}\psi = \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1 + \frac{c^2}{2}\otimes\sigma_3\right)\psi + \mathrm{i}c\sum_{i=1}^{k}\begin{pmatrix}\psi_2(x_i^+) - \psi_2(x_i^-) & 0\\ 0 & \psi_1(x_i^+) - \psi_1(x_i^-)\end{pmatrix}\vec{\delta}_i,$$

in the sense of distributions.

Proof. Let $\psi \in \mathrm{Dom}(H^{\Lambda})$. Then $\psi = \mathrm{e}^{(\sum_{i=1}^{k} H_i(x)\otimes A_i)}\phi$, for $\phi \in H^1(\mathbf{R})\otimes\mathbf{C}^2$, and

$$H^{\Lambda}\psi = \mathrm{e}^{(-\sum_{i=1}^{k} H_i(x)\otimes A_i^*)}\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\phi + \frac{c^2}{2}\otimes\sigma_3\psi.$$

On the other hand, for $\psi \in \mathrm{Dom}(H^{\Lambda})$,

$$\begin{aligned}\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\psi &= \left(-\mathrm{i}c\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_0\otimes\sigma_1\right)\psi\\ &\quad - \mathrm{i}c\sum_{i=1}^{k}\begin{pmatrix}\psi_2(x_i^+) - \psi_2(x_i^-) & 0\\ 0 & \psi_1(x_i^+) - \psi_1(x_i^-)\end{pmatrix}\vec{\delta}_i, \end{aligned} \tag{2}$$

where $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_0$ denotes the derivative away from the centres. Since ψ is absolutely continuous away from the centres, the following holds for x outside any neighbourhood of $\{x_1, \ldots, x_k\}$:

$$\begin{aligned}\left(-\mathrm{i}c\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_0\otimes\sigma_1\right)\psi(x) &= \left(-\mathrm{i}c\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)_0\otimes\sigma_1\right)\mathrm{e}^{(\sum_{i=1}^{k} H_i(x)\otimes A_i)}\phi(x)\\ &= -\mathrm{i}c\sigma_1\,\mathrm{e}^{(\sum_{i=1}^{k} H_i(x)\otimes A_i)}\phi'(x)\\ &= \mathrm{e}^{(-\sum_{i=1}^{k} H_i(x)\otimes A_i^*)}\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\phi(x)\\ &= H^{\Lambda}\psi(x) - \frac{c^2}{2}\otimes\sigma_3\psi(x). \end{aligned} \tag{3}$$

Therefore, combining (2) and (3), we have

$$H^{\Lambda}\psi = \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1 + \frac{c^2}{2}\otimes\sigma_3\right)\psi + \mathrm{i}c\sum_{i=1}^{k}\begin{pmatrix}\psi_2(x_i^+) - \psi_2(x_i^-) & 0\\ 0 & \psi_1(x_i^+) - \psi_1(x_i^-)\end{pmatrix}\vec{\delta}_i,$$

in the sense of distributions. □

Now, we consider the approach in [2], and set $\psi(x_i) = \frac{\psi(x_i^+)+\psi(x_i^-)}{2}$ for $\psi \in \mathrm{Dom}(H^{\Lambda})$. This formal step corresponds to extending the δ_i-functions to the domain of H^{Λ}, which contains functions which may be discontinuous at the centres. Using the boundary conditions $\psi(x_i^+) = \Lambda_i\psi(x_i^-)$, we obtain

$$\begin{aligned}(\Lambda_i + I)\big(\psi(x_i^+) - \psi(x_i^-)\big) &= (\Lambda_i + I)\big(I - \Lambda_i{}^{-1}\big)\psi(x_i^+)\\ &= \big(\Lambda_i - \Lambda_i{}^{-1}\big)\psi(x_i^+)\\ &= 2(\Lambda_i - I)\psi(x_i),\end{aligned}$$

since $2\psi(x_i) = \big(I - \Lambda_i{}^{-1}\big)\psi(x_i^+)$. In the event $\Lambda_i + I$ is invertible, we have, from lemma 1, that

$$H^{\Lambda}\psi = \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x}\otimes\sigma_1\right)\psi + \left(\frac{c^2}{2}\otimes\sigma_3\right)\psi + 2\mathrm{i}c\sum_{i=1}^{k}\sigma_1\frac{\Lambda_i - I}{\Lambda_i + I}\vec{\delta}_i\psi$$

in the sense of distributions, for all $\psi \in \mathrm{Dom}(H^{\Lambda})$.

Remark. For the class of Λ under consideration, namely those satisfying (1), $\Lambda_i + I$ need not be invertible. A trivial example is provided by $\Lambda_i = -I$. In the event $\Lambda_i + I$ is invertible, set $V_i = 2\mathrm{i}c\sigma_1\frac{\Lambda_i - I}{\Lambda_i + I}$. Then V_i is a self-adjoint matrix, and we have the following approximation result:

Theorem 2 (cf [8, theorem 2])**.** *Let $\Lambda_i, i = 1, \ldots, k$, be 2×2 complex-valued matrices satisfying (1), and assume that $\Lambda_i + I$ is invertible. Also, let $\Lambda_i = \exp(A_i)$, $V_i = 2\mathrm{i}c\sigma_1\frac{\Lambda_i - I}{\Lambda_i + I}$, and $H_{i,n} \to H_i$. Then for $\psi \in \mathrm{Dom}(H^\Lambda)$, there exists $\{\psi_n\} \subset H^1(\mathbf{R}) \otimes \mathbf{C}^2$ such that $\psi_n \to \psi$ and $H^{\Lambda_n}\psi_n \to H^\Lambda\psi$ in $L^2(\mathbf{R}) \otimes \mathbf{C}^2$, and for all $\phi \in L^2(\mathbf{R}) \otimes \mathbf{C}^2$,*

$$\left\langle \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\right)\psi + \left(\frac{c^2}{2} \otimes \sigma_3\right)\psi + \sum_{i=1}^{k} V_i \vec{\delta}_i \psi, \phi \right\rangle$$
$$= \lim_{n\to\infty} \left\langle \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3 + \mathrm{i}c\sum_{i=1}^{k} H'_{i,n}(x) \otimes \sigma_1 A_i\right)\psi_n, \phi \right\rangle.$$

The subject of approximation of relativistic delta potentials has been the source of some confusion that has been clarified in several sources (cf [8, 9, 17]). In [9], we showed that the closed-form definitions used here for H^Λ result in straightforward limiting procedures in which the question of renormalization does not arise. On the other hand, when we attempt to formalize the finite-rank perturbations of the free Dirac operator in the way described above (and in [2]), then there is in some instances renormalization when the limit is taken. In fact, from theorems 1 and 2, we see easily that renormalization occurs in all cases *except* when $A_i = 2\frac{\mathrm{e}^{A_i} - I}{\mathrm{e}^{A_i} + I}$, $i = 1, \ldots, k$ (assuming the inverses exist). In the following, we determine precisely those cases for which this condition holds.

Proposition 1 (cf [8, proposition 1])**.** *Let A be a 2×2 complex matrix for which $\mathrm{e}^A + I$ is invertible. Then*

$$A = 2\frac{\mathrm{e}^A - I}{\mathrm{e}^A + I}$$

if and only if one of the following holds:

(i) A is nilpotent, or
(ii) the non-zero eigenvalues of A are simple, nondegenerate and purely imaginary. If $\mathrm{i}y$ is an eigenvalue of A, then y is a real solution of

$$\frac{y}{2} = \tan\frac{y}{2}.$$

We conclude by considering some concrete examples. Returning to the case of Λ_3 discussed earlier, we have that for finitely many point interactions centred at $\{x_1, \ldots, x_k\}$,

$$H^{\Lambda_3}\psi = \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\right)\psi + \left(\frac{c^2}{2} \otimes \sigma_3\right)\psi + \sum_{i=1}^{k} \alpha_i \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{\delta}_i \psi$$
$$= \lim_{n\to\infty} \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3\right)\psi + \mathrm{i}c\sum_{i=1}^{k} H'_{i,n}(x) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\psi$$

in the sense described in theorem 2. Consequently, there is in this case no renormalization. On the other hand, in the case of $\Lambda_1 = \exp(-\mathrm{i}\theta_i \otimes \sigma_1)$,

$$\begin{aligned} H^{\Lambda_1}\psi &= \left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1\right)\psi + \left(\frac{c^2}{2} \otimes \sigma_3\right)\psi + \sum_{i=1}^{k}\left(2c\tan\frac{\theta_i}{2} \otimes I\right)\vec{\delta}_i\psi \\ &= \lim_{n\to\infty}\left(-\mathrm{i}c\frac{\mathrm{d}}{\mathrm{d}x} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3\right)\psi + \sum_{i=1}^{k}(c\theta_i H'_{i,n}(x) \otimes I)\psi. \end{aligned}$$

In this case, renormalization occurs in all cases *except* those for which $\frac{\theta_i}{2} = \tan\frac{\theta_i}{2}$. Finally, we note that we may have different point interactions at the distinct centres if, for example, we have a combination of Λ_1 and $\Lambda_5 = \exp(-\mathrm{i}\theta \otimes I)$, so that the corresponding A $(A_1 = -\mathrm{i}\theta \otimes \sigma_1, A_5 = -\mathrm{i}\theta \otimes I)$ commute.

References

[1] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (New York: Springer)

[2] Albeverio S and Kurasov P 1997 Rank one perturbations of not semibounded operators *Integr. Equ. Oper. Theory* **27** 379–400

[3] Albeverio S, Koshmanenko V, Kurasov P and Nizhnik L 2002 On approximations of rank one $\mathcal{H}_{-2}$-perturbations *Proc. Am. Math. Soc.* **131** 1443–52

[4] Benvegnú S and Dąbrowski L 1994 Relativistic point interactions *Lett. Math. Phys.* **30** 159–67

[5] Chernoff P and Hughes R 1993 A new class of point interactions in one dimension *J. Funct. Anal.* **111** 97–117

[6] Exner P and Šeba P 1988 Mathematical models for quantum point-contact spectroscopy *Czech. J. Phys.* B **38** 1–11

[7] Huddell W and Hughes R 2003 Smooth approximation of singular perturbations of the Laplacian *J. Math. Anal. Appl.* **282** 512–30

[8] Hughes R 1999 Finite rank perturbations of the Dirac operator *J. Math. Anal. Appl.* **238** 67–81

[9] Hughes R 1997 Relativistic point interactions: approximation by smooth potentials *Rep. Math. Phys.* **39** 425–32

[10] Hille E and Phillips R 1957 *American Mathematical Society Colloquium Publications Volume XXXI Functional Analysis and Semigroups* (Providence, RI: American Mathematical Society)

[11] Koshmanenko V 1991 Toward the rank-one singular perturbations theory of self-adjoint operators *Ukr. Math. J.* **43** 1559–66

[12] Kronig R and Penney W 1931 Quantum mechanics of electrons in crystal lattices *Proc. R. Soc. Lond.* A **130** 499–513

[13] Kurasov P 1996 Distribution theory for the discontinuous test functions and differential operators with the generalized coefficients *J. Math. Anal. Appl.* **201** 297–323

[14] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics I. Functional Analysis* (New York: Academic)

[15] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics II. Fourier Analysis and Self-Adjointness* (New York: Academic)

[16] Šeba P 1986 The generalized point interaction in one dimension *Czech. J. Phys.* **36** 667–73

[17] Šeba P 1989 Klein's paradox and the relativistic point interaction *Lett. Math. Phys.* **18** 77–86